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The whole number of ways the three points can be taken is $\frac{81\sqrt{3}}{64}a^6$. Doubling, since the halves are interchangeable, we get for average area of triangle:

$$\triangle=\frac{128}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\int_{-z'}^{t}Adz+\int_t^{z'}A_1dz\right]dudvdwdxdy$$

$$=\frac{64}{81\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\int_{-y'}^{y'}\left[\tfrac{1}{3}(a\sqrt{3}-w)^2+\left(y-\frac{(y-x)(v+w)}{u+v}\right)^2\right]$$

$$\times(u+v)dudvdwdxdy$$

$$=(\tfrac{2}{3})^7\frac{1}{a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\int_{-x'}^{x'}\left[3(a\sqrt{3}-w)^2(a\sqrt{3}-v)(u+v)\right.$$

$$\left.+\frac{(a\sqrt{3}-v)^3(u-w)^2+9x^2(a\sqrt{3}-v)(v+w)^2}{u+v}\right]dudvdwdx$$

$$=(\tfrac{2}{3})^8\frac{\sqrt{3}}{a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^u\left[3(a\sqrt{3}-u)(a\sqrt{3}-v)(a\sqrt{3}-w)^2(u+v)\right.$$

$$\left.+\frac{(a\sqrt{3}-u)(a\sqrt{3}-v)^3(u-w)^2+(a\sqrt{3}-u)^3(a\sqrt{3}-v)(v+w)^2}{u+v}\right]dudvdw$$

$$=(\tfrac{2}{3})^8\frac{1}{\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{3})}\int_0^{\frac{1}{2}(a\sqrt{3})}\left[9a^3\sqrt{3}(a\sqrt{3}-u)(a\sqrt{3}-v)(u+v)\right.$$

$$-3(a\sqrt{3}-u)^4(a\sqrt{3}-v)(u+v)+(a\sqrt{3}-u)^3(a\sqrt{3}-v)(u+v)^2$$

$$\left.+\frac{u^3(a\sqrt{3}-u)(a\sqrt{3}-v)^3-v^3(a\sqrt{3}-u)^3(a\sqrt{3}-v)}{u+v}\right]dudv$$

$$=(\tfrac{2}{3})^5\frac{1}{27\sqrt{3}a^6}\int_0^{\frac{1}{2}(a\sqrt{8})}\left[8\sqrt{3}au^6-42a^2u^5+127\sqrt{3}a^3u^4-480a^4u^3+81\sqrt{3}a^5u^2\right.$$

$$\left.+216a^6u+16u^3(9a^4-u^4)\log\left(\frac{2u+a\sqrt{3}}{2u}\right)\right]du=\frac{1507\sqrt{3}}{11664}a^2.$$

MISCELLANEOUS.

104. Proposed by HARRY S. VANDIVER, Bala, Pa.

A Theorem of Fermat. The area of a right angled triangle with commensurable sides cannot be a square number. [Cf. Chrystal's *Algebra*, Vol. II., page 535.]

II. **Solution by the PROPOSER.**

Analytically expressed, Fermat's assertion is equivalent to the following: No positive integers can be found to satisfy the relation,

$$xy(x^2-y^2)=z^2 \ldots\ldots(1).$$

To prove this suppose that there are three integers, x_0, y_0, z_0, such that

$$x_0y_0(x_0^2-y_0^2)=z_0^2 \ldots\ldots(2).$$

Assume that x_0 and y_0 have a common factor, then we may put $x_0=pa$, and $y_0=pb$. Then substituting, $abp^4(a^2-b^2)=z_0^2$, or by division $ab(a^2-b^2)=\square=u^2$, say (where a and b are prime to each other). Hence it is sufficient to prove the impossibility of (2) only in the case when x_0 and y_0 are prime to each other.

The form ax_1^2 where $a=p\times q\times r\ldots\ldots$ (p, q, r, etc., being primes) and where either a or x_1 may become 1, represents all values of x_0. Putting it in this form, then the quantity a must occur as a factor in $y_0(x_0^2-y_0^2)$ since $x_0y_0(x_0^2-y_0^2)$ is a square number, that is, one of the three following relations must be satisfied:

$$y_0\equiv 0(\text{mod}a)$$
$$x_0-y_0\equiv 0(\text{mod}a)$$
$$x_0+y_0\equiv 0(\text{mod}a)$$

Unless $a=1$, each of these relations show that x_0 and y_0 have a common factor, a result contrary to hypothesis.

Hence $x_0=x_1^2$, and it may be proved in a similar manner that $y_0=y_1^2$.

Substituting these values in (2) we have

$$x_1^2y_1^2(x_1^4-y_1^4)=z_0^2, \text{ whence } x_1^4-y_1^4=\square=z_1^2, \text{ say.}$$

Hence we infer, that if (1) is to be satisfied by integral values of x, y, and z it must be possible to find integers x_1, y_1, z_1 such that

$$x_1^4-y_1^4=z_1^2 \ldots\ldots(3).$$

But it may be shown that there are no such integers x_1, y_1, z_1.

For, consider the expression $x_1^2+y_1^2$. Its general form is bx_2^2 where b is the product of unequal primes, that is,

$$bx_2^2=x_1^2+y_1^2.$$

From this, $\quad x_1^2+y_1^2\equiv 0(\text{mod}b)$

and from (3), $\quad x_1^2-y_1^2\equiv 0(\text{mod}b)$

whence $\quad 2x_1^2\equiv 0(\text{mod}b)$

and $\quad 2y_1^2\equiv 0(\text{mod}b)$

and since x_1 and y_1 are prime to each other, we see that b must have either of the values 1 and 2; that is, either of the two following sets of relations must be satisfied:

$$1\text{st}\begin{cases} x_1^2+y_1^2=x_2^2 \\ x_1^2-y_1^2=y_1^2 \end{cases} \qquad 2\text{nd}\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

Take the first set, writing it as follows:

$$x_1^2-y_1^2=y_2^2 \ldots\ldots(4),$$
$$x_2^2-y_1^2=x_1^2 \ldots\ldots(5).$$

Then proceeding as we did with $x_1^4-y_1^4=z_1^2$, it can be shown that two of the following four sets must hold:

$$\mathrm{I}\begin{cases} x_1+y_1=m^2 \\ x_1-y_1=n^2 \end{cases} \qquad \mathrm{II}\begin{cases} x_2+y_1=r^2 \\ x_2-y_1=s^2 \end{cases} \quad (rs=x_1)$$

$$\mathrm{III}\begin{cases} x_1+y_1=2m^2 \\ x_1-y_1=2n^2 \end{cases} \qquad \mathrm{IV}\begin{cases} x_2+y_1=2r^2 \\ x_2-y_1=2s^2 \end{cases}$$

From (4) and (5) it will be seen that x_1 and x_2 are odd and y_1 is even. Therefore sets III and IV cannot be true. Using the first two and transforming

$$x_1=\frac{m^2+n^2}{2}=rs, \text{ or } y_1=\frac{m^2-n^2}{2}=\frac{r^2-s^2}{2}$$

$$\text{or } m^2+n^2=2rs \ldots\ldots(6),$$
$$m^2-n^2=r^2-s^2 \ldots\ldots(7).$$

Then it is *necessary* that there be integers a, b, l, k such that $r-s=ak$, $r+s=bl$ (any of the quantities a, b, l, k may be $=1$).

Then from (7), $m-n=ab$, $m+n=kl$.

Substituting in (6) the values found for r, s, m, and n from these relations, and reducing,

$$a^2b^2+k^2l^2+a^2k^2-b^2l^2=0,$$

whence $\qquad a^2(b^2+k^2)=l^2(b^2-k^2),$

and therefore $\qquad b^4-k^4=\square \ldots\ldots(8).$

Now since $x_1=\dfrac{m^2+n^2}{2}$, and also $ab=m-n$, and $\dfrac{m^2+n^2}{2}>m-n$,

we have $\qquad x_1>ab,$

whence $\qquad x_1>b.$

Hence, assuming $x_1^2+y_1^2=x_2^2$ and $x_1^2-y_1^2=y_2^2$
we have shown that if (3) is satisfied by x_1, y_1, z_1, then it follows that there is a relation $b^4-k^4=\square$ where $b<x_1$. Using the second assumption, and the only other possible one, namely,

$$\begin{cases} x_1^2+y_1^2=2x_2^2 \\ x_1^2-y_1^2=2y_2^2 \end{cases}$$

we obtain, by addition and subtraction,

$$x_2^2+y_2^2=x_1^2 ......(9),$$
$$x_2^2-y_2^2=y_1^2 ......(10),$$

and multiplying, $x_2^4-y_2^4=\square$. But (9) gives $x_2<x_1$.

Hence it has been shown that using this assumption, we can obtain from (3) a relation $x_2^4-y_2^4=\square$ such that $x_2<x_1$. Therefore, if (3) holds it is *always* possible to find a relation $x_2^4-y_2^4=z_2^2$ such that $x_2<x_1$.

From $x_2^4-y_2^4=z_2^2$ it is then possible to find $x_3^4-y_3^4=z_3^2$ where $x_3<x_2$, and so we can get any number of equations, of the type, $x_n^4-y_n^4=z_n^2$, where

$$x_n<_{n-1}<x_{n-2}......<x_3<x_2<x_1.$$

Since x_n is always positive, by taking n sufficiently large, there is obtained $1-y_n^4=z_n^2$, which is impossible for positive integral values of y_n and z_n.

Hence the impossibility of $x_1^4-y_1^4=z_1^2$ has been completely established, and therefore the fact that $xy(x^2-y^2)=z^2$ cannot be satisfied, follows.

105. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

If the refractive index of a medium at any point be $\mu=x$, prove that the path of the ray will be the curve $\dfrac{2x}{a}=\dfrac{c}{a}\varepsilon^{y/a}+\dfrac{a}{c}\varepsilon^{-(y/a)}$, a and c being constants.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy. Ohio University, Athens, O.

Let (x, y) be any point in the path; $\mu=kx$, the index of refraction; and $p=dy/dx$. The differential equation to the path is

$$\frac{dp/dx}{1+p^2}=\frac{1}{\mu}\left(\frac{d\mu}{dy}-\frac{d\mu}{dx}\frac{dy}{dx}\right)......(1).$$

$d\mu/dx=k$, and (1) reduces to

$$\frac{dp}{p(1+p^2)}=-\frac{dx}{x}......(2).$$